$\underline{1.5}$: $\Gamma < \mathrm{Isom}(\mathbb{R}^2)$ gen. da f & g

Ogni el. di Γ si scrive come $\boxed{f^h \cdot g^k}(x,y) = (\quad , \quad)$

$U \subseteq \mathbb{R}^3$ aperto

$$\begin{tikzcd}
\mathcal{C}^\infty(U) \arrow[r, "\nabla"] \arrow[d, phantom, "\shortparallel"] & \mathfrak{X}(U) \arrow[r, "\mathrm{rot}"] \arrow[d, phantom, "\cong"] & \mathfrak{X}(U) \arrow[r, "\mathrm{div}"] \arrow[d, phantom, "\cong"] & \mathcal{C}^\infty(U) \\
\Omega^0(U) \arrow[r, "d"] & \Omega^1(U) \arrow[r, "d"] & \Omega^2(U) \arrow[ur, "d"] &
\end{tikzcd}$$

$X = (X^1, X^2, X^3)$

$\downarrow$

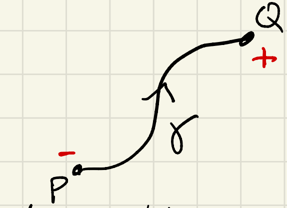
$X^1 dx_1 + X^2 dx_2 + X^3 dx_3$

$\longrightarrow X^1 dx_2 \wedge dx_3 + X^2 dx_3 \wedge dx_1 + X^3 dx_1 \wedge dx_2$

$d^2 = 0 \qquad \mathrm{rot}\,\nabla = 0 \qquad \mathrm{div}\,\mathrm{rot} = 0$

Stokes:

- $\gamma \subseteq \mathbb{R}^2$ curva



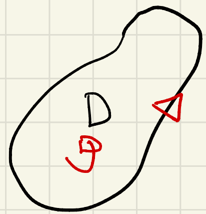
1-varietà con ∂

$f \in C^\infty(U)$

$df \in \Omega^1(U)$

$$\int_\gamma df = \int_{\partial\gamma} f = f(Q) - f(P)$$

- $+/-$

- 

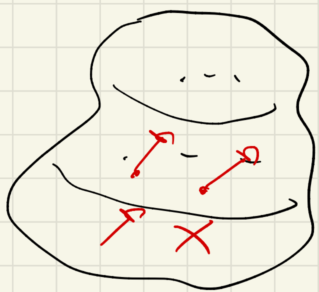
$\mathbb{R}^2$

$\omega \in \Omega^1(D)$

$$\int_D d\omega = \int_{\partial D} \omega$$

$\omega = a\,dx + b\,dy$

Gauss-Green

- 

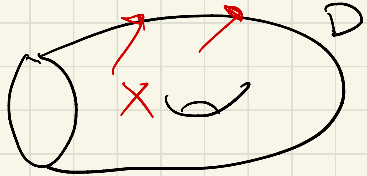
$\mathbb{R}^3$

D

$$\int_D \operatorname{div} X = \int_{\partial D} X \cdot \vec{n}$$

Divergenza

$\odot$



$\mathbb{R}^3$

Stokes

$$\int_D \operatorname{rot} X \cdot \vec{n} = \int_{\partial D} X \cdot \vec{n}$$

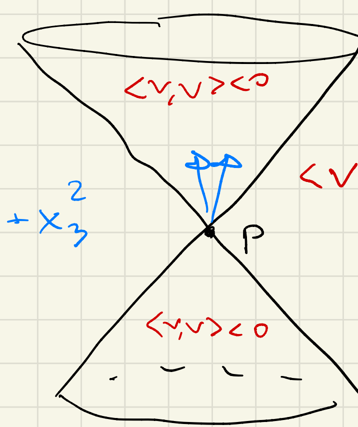
ELETTROMAGNETISMO

Spazio di Minkowski: $\mathbb{R}^4 = \mathbb{R}^{1,3}$ dotato di

un tensore metrico
segnatura $(3,1)$

$$\begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$p \in \mathbb{R}^4 \quad T_p\mathbb{R}^4 = \mathbb{R}^4$



$\langle v, v \rangle > 0$
SPAZIO

$$0 = -c^2 x_0^2 + x_1^2 + x_2^2 + x_3^2$$

Campo elettromagnetico:

$F \in \Omega^2(\mathbb{R}^{1,3})$

$\vec{E} = (E_1, E_2, E_3)$ $\qquad$ $\vec{B} = (B_1, B_2, B_3)$

$$F(x_0, x_1, x_2, x_3) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

<u>Eq di Maxwell</u>: $\boxed{dF = 0}$ $\qquad$ $\boxed{\delta F = J}$ $\qquad$ $J = \begin{pmatrix} -\rho \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}$

$*$ di Hodge: $\boxed{\Omega^k(M) \xrightarrow{\sim} \Omega^{n-k}(M)}$ $\qquad$ $*: \Lambda^k V \xrightarrow{\sim} \Lambda^{n-k}(V)$

M n-varietà con tensore metrico $\qquad$ $\dim V = n$

$$\delta = (-1)^k *^{-1} d \, * : \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\begin{tikzcd} \Omega^k(M) \arrow[r] \arrow[d, "*"] & \Omega^{k-1}(M) \\ \Omega^{n-k}(M) \arrow[r, "d"] & \Omega^{n-k+1}(M) \arrow[u, "*^{-1}"'] \end{tikzcd}$$

CODIFFERENZIALE

<u>Es</u>: $\delta^2 = 0$

COOMOLOGIA DI DE RHAM

S^4, $S^2 \times S^2$, $\mathbb{CP}^2$ non sono diffeom né omeom né omot. eq.

<u>Def</u>: M varietà (senza bordo per pigrizia)

$\omega \in \Omega^k(M)$ è CHIUSA se $d\omega = 0$

ESATTA se $\exists\, \eta \in \Omega^{k-1}(M)$ t.c. $\omega = d\eta$

$d \circ d = 0 \Rightarrow$ (ω esatta $\Rightarrow \omega$ chiusa)

$Z(M) \subseteq \Omega^k(M)$ $\qquad Z(M) = \{\omega \mid d\omega = 0\}$

$B(M) = \{\omega \text{ esatte}\}$

<u>Oss</u>: Se $k > n$ $\quad \Omega^k(M) = \{e\}$

Se $k=n$: ogni $\omega \in \Omega^n(M)$ è chiusa

Ci sono forme chiuse non esatte:

Es: M^n orientata, compatta ω forma volume $\Rightarrow d\omega = 0$

ω non è esatta! Se $\omega = d\eta$ $\eta \in \Omega^{n-1}(M)$

Stokes: $\int_M d\eta = \int_{\partial M = \emptyset} \eta = 0$ assurdo

Es: $S^1 \times S^1$ $\omega = d\vartheta_1$ è chiusa ma non esatta

$(\vartheta_1, \vartheta_2)$

ϑ_i loc però $d\vartheta_i$ è globale $\vartheta_i \notin C^\infty(S^1 \times S^1)$

$d\vartheta_i \in \Omega^1(S^1 \times S^1)$

$\omega = d\vartheta_1$ loc. è differenziale di una funzione

$d\omega = d\,d\vartheta_1 = 0$ <u>loc.</u>

<u>Es:</u> $C = S^1 \times \{0\}$

P. assurdo : $d\omega_1 = d\eta$

$$\int_C d\eta = \int_C d\vartheta_1 = 2\pi \neq 0$$

$$\partial C = \emptyset$$

$$\parallel$$

$$\int_{\partial C = \emptyset} \eta = 0$$

<u>Oss:</u> $\omega \in \Omega^k(M)$ $\quad S \subseteq M$ sottovarietà orientata cpt

$$\int_S \omega := \int_S i^*\omega \neq 0 \implies \omega \text{ non è esatta}$$

per Stokes

<u>Es:</u> $U = \mathbb{R}^2 \setminus \{0\}$ $\quad (\rho, \vartheta)$ polari $\quad \omega = d\vartheta$ è chiusa ma non esatta

perchè $\int_{S^1} \omega = 2\pi$ (come prima)

$$d\vartheta = \frac{x\,dy - y\,dx}{x^2+y^2}$$

Def: Il K-ESIMO GRUPPO DI COOMOLOGIA DI DE RHAM

$$H^k(M) = Z^k(M) \big/ B^k(M) \quad \text{spazio vettoriale}$$

K-ESIMO NUMERO DI BETTI:

$$b^k(M) = \dim H^k(M)$$

Oss: $b^k(M) = 0 \quad \forall k > n = \dim M$

$$\boxed{H^k_{dR}(M) \cong H^k(M, \mathbb{R})}$$

Def: Se $b^i(M) < +\infty$, $\chi(M) = \sum_{i=0}^{n} (-1)^i b^i(M)$ CARATTERISTICA DI EULERO POINCARÉ

Prop: Se M è connessa, $H^0(M) = \mathbb{R}$

dim:
$$H^0(M) = Z^0(M) / \underbrace{B^0(M)}_{\{0\}}$$

$$Z^0(M) \leq C^\infty(M)$$
$$\| $$
$$\{ f : M \to \mathbb{R} \mid df = 0 \}$$

$$H^0(M) = \{ f : M \to \mathbb{R} \text{ cost.} \} = \mathbb{R}$$

Ex: $df = 0 \iff f$ è loc. cost.

Prop: $M = \bigsqcup_{i \in I} M_i$ connesse

$$\Omega^k(M) = \prod_{i \in I} \Omega^k(M_i)$$

$$\left.\begin{array}{l} Z^k(M) = \prod_{i \in I} Z^k(M_i) \\ B^k(M) = \prod_{i \in I} B^k(M_i) \end{array}\right\} \implies H^k(M) = \prod_{i \in I} H^k(M_i)$$

$$H^0(M) = \mathbb{R}^I \qquad b_0(M) = |I| = \# \text{c.c. di } M$$

Prop: $H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{se } k=0 \\ 0 & \text{se } k>0 \end{cases}$ $\qquad H^k(\{pt\}) = \begin{cases} \mathbb{R} & \text{se } k=0 \\ 0 & \text{se } k>0 \end{cases}$

dim:

$k=0$ OK $\qquad k \geq 2$ OK

$k=1$? $\qquad H^1(\mathbb{R}) = 0 \quad$ (TESI)

$$H^1(\mathbb{R}) = \{\text{1-forme (ch.) su } \mathbb{R}\} \big/ \{\text{1-forme esatte su } \mathbb{R}\}$$

$\omega \in \Omega^1(\mathbb{R}) \qquad \omega(x) = f(x)dx$

Mostro che ω è esatta, cioè $\omega = dF \qquad F \in C^\infty(\mathbb{R})$

$F(x) = \int_0^x f(x)$ funziona

$\square$

Teo: $H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{se } k=0 \\ 0 & \text{se } k>0 \end{cases}$ $\qquad \mathbb{R}^n \sim \{pt\}$

Lemma di Poincaré $\qquad\qquad$ omot. eq.

ALGEBRA

M varietà $\qquad H^k(M)$

Prop: $H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$ è un'algebra con l'operazione $\wedge$

dim:

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M) \quad " \quad " \quad " \quad " \quad " \quad "$$

Devo mostrare che $\wedge$ passa al quoziente.

1) $\omega \in Z^k(M) \qquad \eta \in Z^h(M) \qquad \Rightarrow \omega \wedge \eta \in Z^{k+h}(M)$

$$d(\omega \wedge \eta) = \underbrace{d\omega}_{0} \wedge \eta + (-1)^k \omega \wedge \underbrace{d\eta}_{0} = 0$$

2) $\omega \wedge \eta \qquad \eta' = \eta + d\alpha \qquad [\eta] = [\eta'] \in H^h(M)$

$$\overset{?}{\Rightarrow} [\omega \wedge \eta] = [\omega \wedge \eta']$$

cioè $[\omega \wedge d\alpha] = 0 \iff \omega \wedge d\alpha$ ē esatta

$$\underbrace{d(\omega \wedge \alpha)} = \cancel{\underbrace{d\omega}_{0} \wedge \alpha} + (-1)^k \underbrace{\omega \wedge d\alpha}$$ OK

$\square$

FUNTORIALITÀ

$f: M^m \to N^n$ induce $f^*: \Omega^k(N) \to \Omega^k(M)$

$f^*: Z^k(N) \to Z^k(M)$

$f^*: B^k(N) \to B^k(M)$

perché $f^* d\omega = d f^* \omega$

$\Rightarrow f^*: H^k(N) \to H^k(M)$ ben def